

Background independent quantization and wave propagation

Golam Mortuza Hossain,^{*} Viqar Husain,[†] and Sanjeev S. Seahra[‡]

Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada E3B 5A3

(Dated: June 22, 2009)

We apply a type of background independent “polymer” quantization to a free scalar field in a flat spacetime. Using semi-classical states, we find an effective wave equation that is both nonlinear and Lorentz invariance violating. We solve this equation perturbatively for several cases of physical interest, and show that polymer corrections to solutions of the Klein-Gordon equation depend on the *amplitude* of the field. This leads to an effective dispersion relation that depends on the amplitude, frequency and shape of the wave-packet, and is hence distinct from other modified dispersion relations found in the literature. We also demonstrate that polymer effects tend to accumulate with time for plane-symmetric waveforms. We conclude by discussing the possibility of measuring deviations from the Klein-Gordon equation in particle accelerators or astrophysical observations.

PACS numbers: 04.60.Ds

I. INTRODUCTION

There has been much interest in the past few years on possible quantum gravitational effects on wave propagation. The basic idea is that if the metric is quantized, effective wave dynamics follow from replacing the classical metric in the wave equation by the expectation value of a metric operator in a suitable quantum state. Such a state would provide a “quantum gravity corrected” background spacetime with a built in fundamental discreteness scale. It has been conjectured that this may lead to Lorentz symmetry violating modifications to conventional wave propagation and dispersion relations [1]. Phenomenological effects have been studied by introducing explicit Lorentz violating terms in particle physics models [2] with potentially significant effects [3] that could require fine tuning to agree with some observations.

Several authors have previously considered the effects of dispersion relations incorporating high energy corrections on Hawking radiation [4, 5] and inflationary cosmology [6]. In the studies, the precise form of the corrections was specified on an *ad hoc* basis. It was found that the Hawking effect is not affected by such modifications, whereas in cosmology some classes of dispersion relations do affect the spectrum of quantum fluctuations.

It has been argued that quantum gravity modifications to the group velocity of wave-packets can lead to potentially observable effects on the signal from gamma ray bursts [7]. Observed violations of the GZK cutoff [8] in the spectrum of high energy cosmic rays can also indicate a break-down of Lorentz symmetry at high energy. Attempts have been made to systematically derive constraints on parameters appearing in modified dispersion relations such as

$$E^2 = m^2 + p^2 \left[1 + \zeta \left(\frac{p}{p_{\text{pl}}} \right)^\gamma \right] + \dots \quad (1)$$

where E , m and p denote the energy, mass and momentum respectively; p_{pl} is the Planck momentum and ζ and $\gamma > 0$ are real parameters [9].

It is important to ask which ideas associated with the the construction of a quantum theory of gravity lead to potentially observable effects on wave propagation in the semi-classical regime. The work in this direction utilizes a background independent quantization scheme for the dynamical variables associated with geometry. This approach to quantum theory may be applied to any system, and in particular also to matter degrees of freedom. Indeed for a complete quantization of a gravity-matter system it is natural to apply the same procedure to all degrees of freedom.

Our purpose in this paper is to study the effects of a background independent quantization on wave propagation *without any quantization of geometry*. The approach may be viewed as the study of polymer quantum fields on curved spacetime, which is a general area of interest in its own right.

The simplest case is the massless scalar field on a Minkowski background. The kinematics of this quantization, and its comparison to the usual Fock procedure, has been discussed at a mathematical level [10, 11, 12] using a

^{*}Electronic address: ghossain@unb.ca

[†]Electronic address: vhusain@unb.ca

[‡]Electronic address: sseahra@unb.ca

direct application of the ideas developed in loop quantum gravity (LQG). However, so far there appears to have been relatively little study of possible physical consequences of this approach. This is what we undertake here, but with a number of essential differences in approach on which we will elaborate.

This paper is organized as follows. In §II, we review some aspects of background independent quantization, with emphasis on a new set of basic variables for quantization. (These have application well beyond the scenario addressed here.) We then introduce a class of semi-classical states, and show how these lead to a modified wave equation that is both nonlinear and Lorentz invariance violating. This has obvious and significant consequences for wave propagation, even without considering any quantum gravity effects. It provides a sharp contrast to the linear modifications of the wave equation used in all the previous applications mentioned above. We then describe a perturbative solution scheme for the semi-classical wave equation in §III, and its specialization to situations with planar and spherical symmetry. In §IV we discuss the phenomenological implications of the model in the context of modified dispersion relations, the coherence length of collimated beams (with application to the Large Hadron Collider and high energy cosmic rays), and pseudo-spherical radiation from astrophysical sources (such as gamma-ray bursts). A summary and discussion of our results is in §V.

II. QUANTIZATION AND THE EFFECTIVE WAVE EQUATION

A. Polymer formalism

We consider a massless free scalar field in four spacetime dimensions with the following background metric

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + q_{ab}dx^a dx^b . \quad (2)$$

Here $g_{\mu\nu}$ is the spacetime metric whereas q_{ab} denotes the spatial metric. The canonical phase space variables are (ϕ, P_ϕ) with scalar matter Hamiltonian

$$H = \int d^3x \left(\frac{1}{2\sqrt{q}} P_\phi^2 + \frac{\sqrt{q}}{2} q^{ab} \partial_a \phi \partial_b \phi \right) , \quad (3)$$

where $q = \det(q_{ab})$. The usual wave equation for scalar field follows from this via Hamilton's equations.

Let us consider the functions

$$\phi_f = \int d^3y \sqrt{q} f(y) \phi(y), \quad U_\lambda(P_\phi) = \exp[i\lambda P_\phi / \sqrt{q}] , \quad (4)$$

as the new set of basic variables that are to be realized as operators in the quantum theory. Here $f(x)$ is a scalar and λ is a real constant with the dimension of *length squared* in natural units. The factors of \sqrt{q} are necessary to balance density weights in the integral and in the exponent. (These variables may be viewed as the “dual” [13] to those used in the polymer quantization of a particle system [18, 19] motivated by loop quantum gravity LQG [14, 15, 16, 17], where the exponentiated configuration variable is used as the new variable. The dual for quantum mechanical systems has been discussed in [19].) We note however that there are important differences with the variables used for scalar field quantization in [12], which do not use the available background metric in their definition. Our variables satisfy the canonical Poisson bracket

$$\{\phi_f, U_\lambda(P_\phi(x))\} = i\lambda f(x) U_\lambda(P_\phi(x)) . \quad (5)$$

A localized field may be defined by taking for example $f(x)$ to be a Gaussian

$$G(x, x_k, \beta) = e^{-\beta^2(x-x_k)^2} , \quad (6)$$

which is sharply peaked ($1 \ll \beta^2$) at a point x_k . We assume this in the following and write $\phi_G(x_k) \equiv \phi_k$.

Since the representation of these variables that we will use is based on that for polymer quantum mechanics [18, 19], we briefly review this quantization before generalizing it to field theory. The Hilbert space is the space of almost periodic functions, where a wave function is written as the linear combination

$$\psi(p) = \sum_{k=1}^N c_k e^{ix_k p} = \sum_{k=1}^N c_k \langle p | x_k \rangle . \quad (7)$$

Here, the set of points $\{x_k\}$ is a selection (graph) from the real line. The inner product is

$$\langle x|x' \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T dp e^{-ipx} e^{ipx'} = \delta_{x,x'}, \quad (8)$$

in which plane waves are normalizable (the right hand side is the generalization of the Kronecker delta to continuous indices). On this Hilbert space the configuration and translation operators act as

$$-i \frac{\partial}{\partial p} e^{ipx_k} = x_k e^{ipx_k}, \quad \widehat{e^{i\lambda p}} e^{ipx_k} = e^{ip(x_k + \lambda)}. \quad (9)$$

The momentum operator is not defined on this Hilbert space. Consequently, only the *finite* translation can be realized rather than the *infinitesimal* translation.

The generalization of this quantization to field theory is straightforward. The Poisson bracket of the basic scalar field variables is realized as an operator relation on a Hilbert space with basis states

$$|a_1, a_2, \dots, a_n\rangle \quad (10)$$

where the real numbers a_i represent scalar field values at the spatial lattice points x_i . The inner product is

$$\langle a'_1, a'_2, \dots, a'_n | a_1, a_2, \dots, a_n \rangle = \delta_{a'_1, a_1} \dots \delta_{a'_n, a_n} \quad (11)$$

when two states are associated with the same set of lattice points; if not the inner product is zero. For our purpose it suffices to consider a fixed spatial lattice of the type used in numerical computation. This inner product is background (metric) independent in the same way as for example that for Ising model spins; the difference is that for the latter there is a finite dimensional spin space at each lattice point, rather than a real number representing a field value at each point. In contrast the usual Fock space quantization uses the metric dependent Klein-Gordon inner product.

The configuration and translation operators are defined by the following action:

$$\hat{\phi}_f |a_1, a_2, \dots, a_n\rangle \equiv \sum_i a_i f(x_i) |a_1, a_2, \dots, a_n\rangle, \quad (12)$$

$$\widehat{U}_\lambda(P_\phi(x_k)) |a_1, a_2, \dots, a_n\rangle \equiv |a_1, a_2, \dots, a_k + \lambda, \dots, a_n\rangle. \quad (13)$$

It is readily verified that the commutator of these operators is a faithful realization of the corresponding Poisson bracket. The parameter λ represents the discreteness scale in field configuration space. In this representation the momentum operator does not exist. There is however an alternative λ dependent definition of ‘momentum’ given by

$$\hat{P}_\phi^\lambda(x_k) \equiv \frac{\sqrt{q}}{2i\lambda} \left[\hat{U}_\lambda(x_k) - \hat{U}_\lambda^\dagger(x_k) \right], \quad (14)$$

which can be used to define the Hamiltonian operator.

B. Semi-classical states

Our goal is to obtain an effective Hamiltonian using this quantization procedure. This will be done by using semi-classical Gaussian states peaked at a classical phase space configuration. Such states have been defined for Friedmann-Robertson-Walker cosmology in the same representation [20], and we use those results here with a suitable generalization to field theory.

For definiteness, we consider background spacetime here to be Minkowski spacetime with $\sqrt{q} = 1$. This is also the focus of the present paper. In this spacetime consider a space lattice point x_k and the basis states $|a_k = m\lambda\rangle_k$ at this point, where m is an integer. A semi-classical state at x_k is defined by

$$|P_0, \phi_0\rangle_{x_k}^{\sigma, \lambda} = \frac{1}{C} \sum_{m=-\infty}^{\infty} e^{-(\lambda m)^2 / 2\sigma^2} e^{m\lambda\phi_0} e^{-im\lambda P_0} |m\lambda\rangle_{x_k}. \quad (15)$$

This is a Gaussian state of width σ where the (real) parameters P_0 and ϕ_0 represent field values corresponding to a classical configuration at the point x_k . The normalization constant $C > 0$ is given by the convergent sum

$$C^2 = \sum_{m=-\infty}^{\infty} e^{-\lambda^2 m^2 / \sigma^2} e^{2\phi_0 \lambda m}. \quad (16)$$

Calculation with this state gives the following expectation value [20]

$$\langle \hat{U}_\lambda \rangle = e^{i\lambda P_0(x_k)} e^{-\lambda^2/4\sigma^2} K(\lambda, \sigma, \phi_0), \quad (17)$$

where

$$K(\lambda, \sigma, \phi_0) = \left(\frac{1 + 2 \sum_{m \neq 0} \cos \left[\frac{2\pi m \phi_0 \sigma^2}{\lambda} \left(1 + \frac{\lambda}{2\phi_0 \sigma^2} \right) \right] e^{-\pi^2 m^2 \sigma^2 / \lambda^2}}{1 + 2 \sum_{m \neq 0} \cos \left[\frac{2\pi m \phi_0 \sigma^2}{\lambda} \right] e^{-\pi^2 m^2 \sigma^2 / \lambda^2}} \right). \quad (18)$$

Equation (17) together with the definition (14) gives the expectation value

$$\langle \hat{P}_\phi^\lambda(x_k) \rangle = \frac{\sin[P_0(x_k)\lambda]}{\lambda} e^{-\lambda^2/4\sigma^2} K(\lambda, \sigma, \phi_0). \quad (19)$$

This formula has the limits

$$\lim_{\sigma \rightarrow \infty} \langle \hat{P}_\phi^\lambda(x_k) \rangle = \sin[P_0(x_k)\lambda]/\lambda, \quad (20)$$

$$\lim_{\lambda \rightarrow 0} \langle \hat{P}_\phi^\lambda(x_k) \rangle = P_0(x_k). \quad (21)$$

The first limit *i.e.* ignoring the width corrections of the quantum states, shows that the semi-classical state on the field lattice is peaked at the corresponding phase space value. The second limits shows that in the field continuum limit of the momentum expectation value has the appropriate classical value. This is crucial for the quantization scheme to be a viable quantization method given only *finite* field translation operators exist in the representation we are using.

C. Effective Hamiltonian

We now turn to derive an effective Hamiltonian for the scalar field using the semi-classical states (15). Let $\mathcal{H}(P_\phi(x), \phi(x))$ be the Hamiltonian density and we define the effective Hamiltonian density \mathcal{H}^{eff} by the expectation value

$$\mathcal{H}^{\text{eff}}(P_0(x_k), \phi_0(x_k); \sigma, \lambda) \equiv \frac{1}{2} \langle P_0, \phi_0 | (\hat{P}_\phi^\lambda)^2 + \widehat{\nabla^2 \phi} | P_0, \phi_0 \rangle_{x_k}^{\sigma, \lambda}. \quad (22)$$

The kinetic part of the operator gives

$$\begin{aligned} \langle P_0, \phi_0 | (\hat{P}_\phi^\lambda)^2 | P_0, \phi_0 \rangle_{x_k}^{\sigma, \lambda} &= \frac{1}{\lambda^2} \langle P_0, \phi_0 | \left(2 - \hat{U}_{2\lambda}(P_\phi(x_k)) - \hat{U}_{2\lambda}^\dagger(P_\phi(x_k)) \right) | P_0, \phi_0 \rangle_{x_k}^{\sigma, \lambda} \\ &= \frac{1}{\lambda^2} \sin^2(P_0 \lambda) + \mathcal{O}\left(\frac{1}{\sigma}\right), \end{aligned} \quad (23)$$

where the last term represents the width corrections of the semi-classical states. The scalar gradient operator can be defined by an expression such as

$$\frac{\partial \phi}{\partial x} \rightarrow \left(\frac{1}{\epsilon} \hat{\phi}_{x_k + \epsilon} - \hat{\phi}_{x_k} \right), \quad (24)$$

where $\hat{\phi}_{x_k}$ is defined with a Gaussian peaked at x_k as described above, and ϵ is the spacing of a uniform lattice. (Note that there are two lattices in this formulation, one in field space with spacing λ and the other in physical 3-space with spacing ϵ .) Thus for wavelengths large compared to ϵ the space continuum approximation applies, and we can write

$$\mathcal{H}^{\text{eff}}(x) = \frac{1}{2\lambda^2} \sin^2(P\lambda) + \frac{1}{2} \nabla^2 \phi, \quad (25)$$

where we have dropped the subscripts on P_0 and ϕ_0 . In this expression and what follows, we have neglected corrections due to the finite width of the semi-classical state. That is, we will work in the $\sigma \rightarrow \infty$ limit for the rest of this paper. Finally, the effective Hamiltonian

$$H^{\text{eff}} = \int d^3x \mathcal{H}^{\text{eff}}(P, \phi) \quad (26)$$

gives the modified wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \left[1 - 4\lambda_*^2 \left(\frac{\partial \phi}{\partial t} \right)^2 \right]^{1/2} \nabla^2 \phi = 0, \quad (27)$$

where we have set $\lambda = \lambda_*$ as a fixed “polymerization scale”. The nonlinear term represents the continuum remnant of polymer quantization. This equation is not Lorentz invariant (due to the $\dot{\phi}^2$ term), but is invariant under Galilean and parity ($x \rightarrow -x$, $t \rightarrow -t$) transformations, as well as field reflection, $\phi \rightarrow -\phi$. We note also that the effective Hamiltonian is the integral of local density so local casual evolution via Hamilton’s equations is assured. Indeed the corrections to the Hamiltonian in an expansion in λ_* gives terms that are higher derivative in momenta, which are like higher derivative terms from the lagrangian perspective.

A generalization of this equation to arbitrary space metric q_{ab} with lapse $N = 1$ and shift $N^a = 0$ derived using the same considerations appears in Appendix A.

D. Homogeneous scalar field on a cosmological background

The formalism of polymer quantization presented here can be applied to different background spacetimes. In particular, it is useful to see what happens if we consider a reduction of the theory to the homogeneous spacetime where it is a quantum mechanical system. Consider the Friedmann-Robertson-Walker (FRW) background

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \quad (28)$$

so that $\sqrt{q} = a^3$ where a is the scale factor. The basic observables (4) therefore reduce to

$$\phi_f = V_0 a^3 \phi, \quad U_\lambda = e^{i\lambda P/a^3}, \quad (29)$$

where V_0 is a fiducial volume obtained from the spatial integration, and we have set $f = 1$ (which is natural given that any other non-zero constant can be absorbed in V_0). The algebra obtained from the fundamental bracket (5) gives

$$\{\phi, U_\lambda\} = \frac{i\lambda}{V_0 a^3} U_\lambda. \quad (30)$$

These relations have an interesting feature when compared to the so called “improved dynamics” [21] scenario in loop quantum cosmology (LQC) [22]: The appearance of the a^3 factor in U_λ and the reduced Poisson bracket is natural from the start, given the definition of the basic variables that we are using. This feature leads to the so-called dynamical lattice refinement in cosmology that is necessary to reproduce correct classical limit. In the LQC context on the other hand, there is no natural way to see how these factors can arise from the full theory upon reduction of a holonomy, which has no metric (\sqrt{q}) dependent factors in it.

Using these variables one can study the effective dynamics of a homogeneous polymer scalar field coupled to an FRW background, with rather surprising results: there is singularity avoidance and a built-in inflationary phase, *without any need to quantize gravity* [23]. In this scenario, the non-singular universe is past inflating. This is in contrast to LQC where the singularity avoidance is realized through a quantum *bounce* that occurs when the density of matter reaches the Planck density [21, 24, 25]. Consideration of inhomogeneous fluctuations in this inflating background can provide a probe to study the polymer effects through the power spectrum of cosmological fluctuations. This is presently under investigation.

III. PERTURBATIVE SOLUTIONS

A. General solution

Our goal in this section is to develop perturbative solutions to the equation (27). As a first step, we rewrite the equation in terms of dimensionless coordinates and fields. Noting that the parameter λ_* has dimensions of length squared, we define a mass scale M_* by

$$M_* = 1/\sqrt{\lambda_*}. \quad (31)$$

Using the mass scale M_* , we can define a set of dimensionless quantities as follows

$$\Phi = \phi/M_*, \quad T = M_* t, \quad X^i = M_* x^i. \quad (32)$$

In terms of these variables, the equation (27) becomes

$$\ddot{\Phi} - (1 - 4\dot{\Phi}^2)^{1/2} D^2 \Phi = 0, \quad (33)$$

where an overdot indicates $\partial/\partial T$ and $D_i = \partial/\partial X^i$. Notice that neither λ_* nor M_* appears in this equation. The nonlinear wave equation (33) admits the trivial solution $\Phi = 0$. We construct a perturbative series about this exact solution as follows:

$$\Phi = \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots = \sum_{i=1}^{\infty} \epsilon^i \Phi_i, \quad (34)$$

where $\epsilon \ll 1$ is a small dimensionless parameter. If we put this ansatz into the wave equation, expand in powers of ϵ , and set the coefficient of each power equal to zero, we obtain:

$$\square \Phi_1 = 0, \quad (35a)$$

$$\square \Phi_2 = 0, \quad (35b)$$

$$\square \Phi_3 = 2\dot{\Phi}_1^2 D^2 \Phi_1, \quad (35c)$$

$$\square \Phi_4 = 2\dot{\Phi}_1(\dot{\Phi}_1 D^2 \Phi_2 + 2\dot{\Phi}_2 D^2 \Phi_1), \quad (35d)$$

$$\square \Phi_5 = 2(\dot{\Phi}_1^4 + \dot{\Phi}_2^2 + 2\dot{\Phi}_1 \dot{\Phi}_3) D^2 \Phi_1 + 2\dot{\Phi}_1(\dot{\Phi}_1 D^2 \Phi_3 + 2\dot{\Phi}_2 D^2 \Phi_2), \quad (35e)$$

\vdots

where $\square = -\partial_T^2 + D^2$.

Before moving on, a few comments about these perturbative solutions are in order. First, note that the perturbative equations of motion (35) imply that we may absorb the Φ_2 contribution into Φ_1 by making the substitution $\Phi_1 + \epsilon \Phi_2 \rightarrow \Phi_1$. That is, we can effectively set $\Phi_2 = 0$. This then implies that we can absorb the Φ_4 into Φ_3 via $\Phi_3 = \Phi_3 + \epsilon \Phi_4$, and so on. Thus at the end, we are left with $\Phi_{2i} = 0$ and a perturbative series with only odd powers of ϵ .

Secondly, it is perhaps useful to elaborate on the nature of our perturbative expansion. In field theory, perturbative series solutions often make use of dimensionless factors appearing in the action or equations of motion as expansion parameters. The prototypical example of this is a scalar field with a $V(\phi) = \mu \phi^4$ self-interaction. In such a model, it is natural to develop a perturbative solution in terms of the dimensionless parameter μ ; i.e., $\phi = \sum_n \mu^n \phi_n$. Our situation is somewhat different since the wave equation (33) has no small dimensionless parameters appearing in it. Hence, we instead develop a series solution in terms of the *amplitude* of Φ . Such expansions are familiar from elementary physics; for example, consider the equation of motion of a simple pendulum of length ℓ in a uniform gravitational field g :

$$\ddot{\theta} = -\sin \theta, \quad (36)$$

where the dimensionless time is $T \equiv (g/\ell)^{1/2} t \equiv \omega t$ and θ measures the angular position. Like (33), this equation has no naturally small parameters appearing in it. To solve for θ , we employ an expansion

$$\theta(T) = \sum_{n=1}^{\infty} \epsilon^n \theta_n(T), \quad (37)$$

which leads to equations of motion

$$\ddot{\theta}_1 + \theta_1 = 0, \quad \ddot{\theta}_2 + \theta_2 = 0, \quad \ddot{\theta}_3 + \theta_3 = \frac{1}{6} \theta_1^3, \quad (38)$$

and so on. These are very similar in structure to (35), and we see that θ_1 satisfies the familiar equation of motion for a simple harmonic oscillator. The parameter ϵ has a natural interpretation when we write down the full θ solution

$$\theta = \epsilon \cos(\omega t + \delta) + \dots \quad (39)$$

That is, ϵ is just the dimensionless amplitude of the leading order term, which implies that the perturbative series is only valid for small angle deviations from the vertical. In our perturbative solution of (33) we have precisely the

same interpretation: ϵ is the amplitude of the leading order solution for Φ and our perturbative series is only valid for small amplitude fluctuations.

Finally, we mention that to re-introduce dimensions correctly one must multiplying (34) by M_* :

$$\phi = \phi_{\text{KG}} + \sum_{i=1}^{\infty} \epsilon^{2i} \phi_{2i+1}, \quad \phi_{\text{KG}} = \epsilon M_* \Phi_1, \quad \phi_{2i+1} = \epsilon M_* \Phi_{2i+1}. \quad (40)$$

Note that ϕ_{KG} satisfies the dimensionful Klein-Gordon equation $(-\partial_t^2 + \nabla^2)\phi_{\text{KG}} = 0$.

B. Planar symmetry

We now seek solutions of the perturbative equations of motion (35) with planar symmetry. That is, if we write the dimensionless spatial coordinates as $X^i = (X, Y, Z)$, then all the fields Φ_i depend on T and X only. Defining the retarded and advanced times as

$$U = T - X, \quad V = T + X, \quad (41)$$

respectively, we assume that the leading order contribution Φ_1 is a plane-fronted wave packet propagating in the positive X direction:

$$\Phi_1 = F(T - X) = F(U). \quad (42)$$

Here, F is an arbitrary function. As mentioned above, we can take $\Phi_2 = 0$ without loss of generality. For Φ_3 , we find that (35c) reduces to

$$\frac{\partial^2 \Phi_3}{\partial U \partial V} = -\frac{1}{2} F''(U) [F'(U)]^2, \quad (43)$$

with general solution

$$\Phi_3 = g_L(V) + g_R(U) - \frac{1}{6} V [F'(U)]^3, \quad (44)$$

where g_L and g_R are arbitrary functions. To fix these functions, we need to choose initial conditions. A physically interesting situation involves preparing the total field Φ such that the polymer corrections are zero on some initial hypersurface Σ_0 ; i.e., Φ corresponds to a wave packet that initially evolves according to the Klein-Gordon equation. Obviously, the particular choice of the initial data hypersurface will affect the form of the solution. For simplicity, we choose Σ_0 to be the null hypersurface $V = 0$ and we require

$$\Phi_3|_{V=0} = 0. \quad (45)$$

This means that $g_L(V) = 0$ and $g_R(U) = 0$. Restoring dimensions, we obtain the following solution for ϕ :

$$\phi = M f \left[1 - \frac{v}{6f} \frac{M^2}{M_*^4} \left(\frac{df}{du} \right)^3 + \dots \right], \quad (46)$$

where

$$M = \epsilon M_*, \quad u = t - x, \quad v = t + x, \quad f = f(u) = F(M_* u). \quad (47)$$

Here, M is a mass scale representing the amplitude of the leading order (classical) solution. We now consider a couple of explicit solutions which are of physical interest. If the leading order term in the series (46) represents a Gaussian wave packet $f = e^{-\beta^2 u^2} e^{iku}$ where the width of the wave-packet is $1/\beta$, we obtain

$$\phi = M e^{-\beta^2 u^2} e^{iku} \left[1 + \frac{1}{6} \frac{M^2 k^2}{M_*^4} \left(1 + \frac{2i\beta^2 u}{k} \right)^3 i k v e^{-2\beta^2 u^2} e^{2iku} + \dots \right]. \quad (48)$$

Conversely, if the leading term represents a plane wave $f = e^{iku} = e^{ik(t-x)}$ then we get

$$\phi = M e^{ik(t-x)} \left[1 + \frac{1}{6} \frac{M^2 k^2}{M_*^4} i k (t+x) e^{2ik(t-x)} + \dots \right]. \quad (49)$$

In either case, we see that the polymer corrections are small if

$$kv = k(t+x) \ll \left(\frac{M_\star}{M}\right)^2 \left(\frac{M_\star}{k}\right)^2. \quad (50)$$

Naively, we recover Klein-Gordon solutions in the limit of small amplitudes $M \ll M_\star$ and/or long wavelengths $k \ll M_\star$. However, notice that when $(t+x)$ becomes large, the leading order polymer correction grows to become comparable to the Klein-Gordon contribution. In other words, the perturbative solutions will eventually break down. This implies that if we wait long enough, then the nonlinear quantum effects will affect the wave propagation significantly, irrespective of the values of k or M .

Previously, we considered the initial data hypersurface to be null. We now consider the situation where initial data are specified at the $T = 0$ hypersurface. As earlier, we require “purely Klein-Gordon” data on the initial hypersurface. That is, Φ_3 should satisfy the following conditions

$$\Phi_3|_{T=0} = 0, \quad \dot{\Phi}_3|_{T=0} = 0. \quad (51)$$

Here it is slightly more involved to find explicit expressions for g_L and g_R . In the case where the leading order term represents a plane wave, we get

$$\phi = M e^{ik(t-x)} \left[1 + \frac{i}{3} \frac{k^2 M^2}{M_\star^4} e^{2ik(t-x)} kt \left(1 - \frac{e^{-3ikt} \sin 3kt}{3kt} \right) + \dots \right]. \quad (52)$$

It follows from the expression (52) of perturbative solutions that polymer corrections are small only if

$$kt \ll \left(\frac{M_\star}{M}\right)^2 \left(\frac{M_\star}{k}\right)^2. \quad (53)$$

This is effectively the same criterion as found earlier for the initial condition specified at null hypersurface (45). In particular, we see that after a sufficiently long time the polymer corrections will become non-negligible and the perturbative solutions will break down.

C. Spherical symmetry

It is also interesting to solve the perturbation equations (35) under the assumption of spherical symmetry; i.e., the fields are functions of t and a radial coordinate r . If we assume that the leading order contribution represents an outgoing spherical wave packet, we obtain the following solution:

$$\phi = \frac{M e^{-\beta^2 u^2} e^{iku}}{kr} \left[1 + \frac{2i}{3} \frac{k^2 M^2}{M_\star^4} \left(\frac{1}{kv_0} - \frac{1}{kv} \right) \left(1 + \frac{2i\beta^2 u}{k} \right)^3 e^{-2\beta^2 u^2} e^{2iku} + \dots \right]. \quad (54)$$

Here, $u = t - r$ and $v = t + r$. We have imposed the boundary conditions such that ϕ reduces to the Klein-Gordon solution $\phi = M e^{iku}/kr$ on an initial null hypersurface $v = v_0$. It is interesting to note that for late times (i.e, $v \rightarrow \infty$) the solution reduces to

$$\phi = \frac{M e^{-\beta^2 u^2} e^{iku}}{kr} \left[1 + \frac{2i}{3} \frac{k M^2}{v_0 M_\star^4} \left(1 + \frac{2i\beta^2 u}{k} \right)^3 e^{-2\beta^2 u^2} e^{2iku} + \dots \right], \quad (55)$$

which is actually a solution of the ordinary Klein-Gordon equation $(-\partial_t^2 + \nabla^2)\phi = 0$. In contrast to case of planar waves, the polymer effects on spherical waves gradually decreases as the wave packet travels away from the initial surface. The reason behind this is that the effective amplitude of the Klein-Gordon part of the solution decays as $1/v$ when it is far away from $r = 0$. Since the polymer corrections are explicitly *amplitude dependent*, it follows that we recover Klein-Gordon behaviour as $v \rightarrow \infty$. It may be emphasized that this behaviour however does not mean that the asymptotic form of ϕ exactly matches the Klein-Gordon prediction. In particular, the polymer effects generate a whole series of additional wave packets with frequency $3k, 5k$, etc in addition to the original wave packet with frequency k . The relative amplitude of the “polymer harmonic” with $k_n = (2n+1)k$ will be roughly $(kM^2/v_0 M_\star^4)^n$, where v_0 can be taken as the initial radius of the wave packet. From this, it follows that our perturbative solutions will be valid at infinity if

$$\frac{k}{M_\star} \ll (M_\star v_0) \left(\frac{M_\star}{M}\right)^2. \quad (56)$$

That is, low wavenumber and low amplitude waves are well described by perturbative techniques.

IV. PHENOMENOLOGICAL IMPLICATIONS

In this section, we explore some phenomenological implications of the perturbative solutions of the semi-classical Klein-Gordon equation. We begin by analyzing the propagation of a planar wave-packet and derive its group velocity and an effective “dispersion relation” (which must be interpreted with care). We then examine the conditions under which our perturbative solutions break-down, and hence derive a “coherence length” over which planar waves resemble solutions of the ordinary Klein-Gordon equation. Finally, we consider the spherical wave-packet solutions in the context of polymer modifications to the spectrum of gamma ray bursts.

A. Group velocity of polymer wave-packets and effective dispersion relations

As mentioned earlier, there has been much discussion in the literature concerning “modified dispersion relations” of the form (1), especially in the context of doubly special relativity [26, 27]. Such formulae are consistent with linear wave equations with higher derivatives; for example, an equation of the form $[-\partial_t^2 + \nabla^2 - l_{\text{pl}}^2 \nabla^4]\phi = 0$ leads to the dispersion relation $\omega^2 = k^2(1 + l_{\text{pl}}^2 k^2)$. However, for nonlinear wave equations such as the polymer-corrected Klein-Gordon equation (27), the derivation and interpretation of dispersion relations is more complicated. In this section, we suggest a physically-motivated derivation of an effective dispersion relation based on the propagation of spatially localized wave-packets.

Let us begin by considering the polymer corrected planar wave solution (46). We assume that the leading order Klein-Gordon term in the series represents a spatially localized waveform with characteristic frequency k :

$$f(u) = h(ku)e^{iku}. \quad (57)$$

The real and dimensionless function h defines the envelope of the wave-packet. For simplicity, we restrict ourselves to symmetric wave-packets with $h(\xi) = h(-\xi)$ and define M such that $h(0) = 1$. Also, we assume that h vanishes strongly at infinity such that integrals of the form $\int \xi^n h^2(\xi) d\xi$ are finite. Sufficient conditions for this hold are that h has compact supports or decays exponentially for large arguments. Note that our choice of $f(u)$ means that the Klein-Gordon contribution to the perturbation series $\phi_{\text{KG}} = Mh(ku)e^{iku}$ is self-similar; i.e., ϕ_{KG} is invariant under a rescaling of the coordinate $u \rightarrow \ell u$ if we also rescale the wavenumber $k \rightarrow k/\ell$.

In this subsection, we are interested in the leading order polymer effects, so we truncate the series (46) as follows:

$$\phi \approx Mh(ku)e^{iku} - \frac{v}{6} \frac{M^3}{M_\star^4} \left\{ \frac{d}{du} [h(ku)e^{iku}] \right\}^3. \quad (58)$$

We define the position of the packet at a given time t by the average value of x :

$$x_{\text{avg}} = \frac{\int_{-\infty}^{\infty} dx \, x \phi^* \phi}{\int_{-\infty}^{\infty} dx \, \phi^* \phi} = t - \frac{\int du \, u \phi^* \phi}{\int du \phi^* \phi}, \quad (59)$$

where we have made use of $u = t - x$. In general, x_{avg} will depend on time and the wavenumber k of the packet. It is easy to confirm that in the Klein-Gordon limit $M_\star \rightarrow \infty$ we get $x_{\text{avg}} = t$; i.e. the pulse propagates at the speed of light.

To calculate x_{avg} in general, it is useful to define

$$\mathcal{Q}(\xi) = h(\xi)e^{i\xi} \left\{ \frac{d}{d\xi} [h(\xi)e^{-i\xi}] \right\}^3 + \text{c. c.} \quad (60)$$

From the fact that h is even, it follows that \mathcal{Q} is an odd function of ξ . We also define the dimensionless moments

$$\mu_n \equiv \int_{-\infty}^{\infty} d\xi \, \xi^n h^2(\xi), \quad \nu_n \equiv \int_{-\infty}^{\infty} d\xi \, \xi^n \mathcal{Q}(\xi), \quad (61)$$

with $n = 0, 1, 2, \dots$. From the parity of h and \mathcal{Q} , these satisfy $\mu_{2n+1} = 0$ and $\nu_{2n} = 0$. Using these definitions and the fact that $v = 2t - u$, we find that

$$\phi\phi^* = M^2 |h(ku)|^2 - \frac{k^3(2t-u)}{6} \left(\frac{M}{M_\star} \right)^4 \mathcal{Q}(ku) + \mathcal{O}\left(\frac{1}{M_\star^8}\right), \quad (62)$$

which leads to

$$\int u^n \phi^* \phi du = \frac{M^2}{k^{n+1}} \mu_n - \frac{t}{3k^{n-2}} \left(\frac{M}{M_\star} \right)^4 \nu_n + \frac{1}{6k^{n-1}} \left(\frac{M}{M_\star} \right)^4 \nu_{n+1} + \mathcal{O} \left(\frac{1}{M_\star^8} \right). \quad (63)$$

This gives the effective position of the pulse as a function of time:

$$x_{\text{avg}} = t \left(1 + \frac{M^2 k^2}{3M_\star^4} \frac{\nu_1}{\mu_0} \right) + \mathcal{O} \left(\frac{1}{M_\star^8} \right). \quad (64)$$

From this expression, we see that the polymer wave-packet travels with a group velocity different from unity

$$v_g = \frac{dx_{\text{avg}}}{dt} = 1 + \frac{M^2 k^2}{M_\star^4} \mathcal{F}_h + \mathcal{O} \left(\frac{1}{M_\star^8} \right), \quad \mathcal{F}_h \equiv \frac{\nu_1}{3\mu_0}. \quad (65)$$

Here, \mathcal{F}_h is a dimensionless form factor that depends on the wave-packet profile h only. For example, if h is a Gaussian we obtain

$$h(\xi) = e^{-\xi^2/\alpha^2} \Rightarrow \mathcal{F}_h = \frac{\sqrt{2}}{96} \left(\frac{\alpha^2 - 12}{\alpha^2} \right) e^{-\alpha^2/4}. \quad (66)$$

We see from (65) that the polymer correction to the group velocity depends not only of the wavenumber k , but also on the Klein-Gordon amplitude M and the shape of the pulse \mathcal{F}_h . This formula for v_g can be physically interpreted by noting that the characteristic energy density associated with the Klein-Gordon part of the wave-packet is

$$\rho_{\text{KG}} = \frac{1}{2} [(\partial_t \phi_{\text{KG}})^2 + (\nabla \phi_{\text{KG}})^2] \sim M^2 k^2. \quad (67)$$

Then we see that

$$v_g - 1 = \mathcal{O} \left(\frac{\rho_{\text{KG}}}{M_\star^4} \mathcal{F}_h \right). \quad (68)$$

That is, the polymer correction to the group velocity scales like the density of the wave-packet normalized by the “polymer density” M_\star^4 . This is somewhat akin to the situation in loop quantum cosmology, where polymer corrections to the Friedmann equation scale like the density normalized by the gravitational polymer scale M_{pl}^4 . Previously in the literature, the astrophysical and experimental consequences of modified group velocities have been derived by considering corrections that depend only on k [7]. This assumption was motivated by *ad hoc* dispersion relations of the form (1). Here, we see that the corrections can depend on the amplitude and shape of the wave-packet as well. This crucial difference is due to the fact that we are dealing with a nonlinear semi-classical wave equation. It is worthwhile stressing that the above group velocity has been derived, not postulated as in previous work.

Given the above expression for the group velocity, one may be tempted to derive an effective dispersion relation by setting $v_g = d\omega/dk$ and integrating with respect to k . This can be done, and we find

$$\omega^2 = k^2 \left[1 + \frac{2}{3} \frac{M^2 k^2}{M_\star^4} \mathcal{F}_h + \mathcal{O} \left(\frac{1}{M_\star^8} \right) \right]. \quad (69)$$

On first glance, this seems to be of the same form as (1) with $\gamma = 2$. But we again note that corrections depend on the amplitude M and shape \mathcal{F}_h of the original wave-packet. We comment that amplitude dependent dispersion relations are a common consequence of nonlinear wave equations; for example, they arise in the theory of deep water surface waves [28].

We conclude by noting that significant caution is warranted if we try to interpret (69) as the dispersion relation for some linear wave equation with higher derivative terms. For example, the fact that $d^2\omega/dk^2 \neq 0$ would naïvely imply that our wave-packet is spreading as time progresses. But an explicit calculation of the effective width of the pulse gives

$$\Delta x^2 \equiv \frac{\int_{-\infty}^{\infty} dx x^2 \phi^* \phi}{\int_{-\infty}^{\infty} dx \phi^* \phi} - x_{\text{avg}}^2 = \frac{\mu_2}{k^2 \mu_0} + \frac{1}{6M^2 \mu_0} \left(\frac{M}{M_\star} \right)^4 \left(\nu_3 - \frac{\mu_2 \nu_1}{\mu_0} \right) + \mathcal{O} \left(\frac{1}{M_\star^8} \right). \quad (70)$$

Note this is time independent to leading order in M_\star . So although the dispersion relation insinuates that the packet is spreading, a direct calculation shows that this is not the case to leading order. Fundamentally, simple dispersion relations of the form (1) are not capable of capturing all of the interesting dynamics associated with our nonlinear wave equation (27).

B. Effective coherence length for planar waves

In this subsection, we discuss the break-down of the perturbative solutions to the polymerized wave equation and the associated observational effects. As we have seen in Sections III B and III C, the polymer quantum effects accumulate or dissipate with time for planar and spherical waves, respectively. So, to maximize our chances of seeing deviations from conventional theory we should look at highly collimated beams of radiation. To quantify how big polymer effects are in such beams, we rewrite our polymer-corrected plane wave solution (49) as

$$\phi = \phi_{\text{KG}} \left[1 + \frac{i}{6} \left(\frac{t+x}{d_{\text{poly}}} \right) e^{2ik(t-x)} + \dots \right], \quad (71)$$

where

$$\phi_{\text{KG}} = M e^{ik(t-x)}, \quad d_{\text{poly}} = t_{\text{poly}} = \frac{1}{k} \left(\frac{M_\star}{M} \right)^2 \left(\frac{M_\star}{k} \right)^2. \quad (72)$$

As we have mentioned before, the magnitude of the polymer corrections inside the square brackets becomes larger as time goes on, and our perturbative solutions are valid for

$$t \ll t_{\text{poly}}. \quad (73)$$

That is, when this condition is satisfied the approximation $\phi \approx \phi_{\text{KG}}$ holds. We note that we can obtain a very similar expression to (71) if ϕ_{KG} is a wave packet whose width is much larger than $1/k$. In that case, the validity of the classical solution demands that the distance d traveled by the wave packet satisfy

$$d \ll d_{\text{poly}}. \quad (74)$$

It is useful to think of t_{poly} and d_{poly} as a “coherence time” and “coherence length”, respectively, because they represent the time or distance over which the polymer corrections to a classical wave packet remain small.

More practical expressions for t_{poly} and d_{poly} come from noting that the luminosity (particles per unit area per unit time) associated with the Klein-Gordon part of the beam is

$$L_{\text{KG}} = \frac{\rho_{\text{KG}}}{k} = \frac{1}{2k} [(\partial_t \phi_{\text{KG}})^2 + (\nabla \phi_{\text{KG}})^2] \sim M^2 k. \quad (75)$$

Restoring conventional CGS units, this yields

$$d_{\text{poly}} = l_{\text{pl}} \left(\frac{M_\star}{M_{\text{pl}}} \right)^4 \left(\frac{E_{\text{KG}}}{E_{\text{pl}}} \right)^{-2} \left(\frac{L_{\text{KG}}}{L_{\text{pl}}} \right)^{-1}, \quad L_{\text{pl}} = \frac{1}{l_{\text{pl}}^2 t_{\text{pl}}} = 7 \times 10^{108} \text{ cm}^{-2} \text{ sec}^{-1}, \quad (76)$$

where $E_{\text{KG}} = \hbar k/c$ is the energy per particle in the beam and L_{pl} is the Planck luminosity. From this, we see that the coherence length is minimized for high-energy and large luminosity beams.

One of the most luminous artificial beams ever constructed is in the Large Hadron Collider (LHC) [29]. In that device, the proton luminosity is $L_{\text{KG}} \sim 10^{34} \text{ cm}^{-2} \text{ sec}^{-1}$ at an energy of $E_{\text{KG}} \sim 7 \text{ TeV}$. This corresponds to

$$d_{\text{poly}} \sim \left(\frac{M_\star}{M_{\text{pl}}} \right)^4 10^{45} \text{ Gpc}. \quad (77)$$

The coherence length is longer if we consider the most energetic cosmic rays: The Pierre Auger Observatory observed 561 particles of energy $\sim 10^{10} \text{ GeV}$ with an integrated exposure of $\sim 10^5 \text{ km}^2 \text{ yr}$ [30], which corresponds to

$$d_{\text{poly}} \sim \left(\frac{M_\star}{M_{\text{pl}}} \right)^4 10^{84} \text{ Gpc}. \quad (78)$$

Unless $M_\star \ll M_{\text{pl}}$, both of these distances are many orders of magnitude larger than the size of the observable universe (which is $\sim 3 \text{ Gpc}$). Hence, if the polymer scale is of the same order as the Planck scale, the Klein-Gordon equation gives an excellent description of the physics in both scenarios, and polymer effects will be very hard to observe. However, we should point out that there is no *a priori* reason to take $M_\star \sim M_{\text{pl}}$ in this model. Indeed if we take $M_\star = 5 \text{ TeV}$, we find that $t_{\text{poly}} = 1 \text{ sec}$ in the LHC. That is, polymer effects would completely dominate the beam after only one second. Since we are unaware of any exotic beam behaviour in the LHC, this can serve as an effective lower bound: $M_\star \gtrsim 5 \text{ TeV}$.

C. Spectral modifications to gamma ray bursts

We conclude this section by considering the spherical wave packets of Section III C. We can use these to model the radiation emitted from a gamma-ray burst (GRB) if we assume that the energy emitted by the progenitor is isotropic and we can approximate the electromagnetic wave dynamics using a scalar field. Both assumptions should be sufficient for the order of magnitude estimates we seek here.

We imagine the following situation: The gamma-ray progenitor is located a distance of d away from the earth. At $t = t_0$ a spherical pulse of radiation of radius $r_0 = t_0$ is emitted from this progenitor. The initial pulse profile is assumed to be Gaussian:

$$\phi \approx \phi_{\text{KG}} = \frac{M e^{-\beta^2(t-r)^2} e^{ik(t-r)}}{kr}, \quad (t \approx t_0, r \approx r_0). \quad (79)$$

Now, the signal $S(\tau)$ seen by an earth based detector will just be the polymer solution (54) evaluated at $r = d$:

$$S(\tau) \equiv \phi|_{r=d} = \frac{M e^{-\beta^2\tau^2} e^{ik\tau}}{kd} \left[1 + \frac{i}{3} \frac{kM^2}{M_\star^4 r_0} \left(1 + \frac{2i\beta^2\tau}{k} \right)^3 e^{-2\beta^2\tau^2} e^{2ik\tau} + \dots \right], \quad (80)$$

with $\tau = t - d$ and assuming $d \gg r_0$. The Fourier transform of this signal is

$$S(\omega) = \frac{1}{\sqrt{2\pi}} \int d\tau e^{-i\omega\tau} S(\tau) = S_1 e^{-(\omega-k)^2/4\beta^2} + iS_3 \left(\frac{\omega}{3k} \right)^3 \left[1 - \frac{18\beta^2}{\omega^2} \right] e^{-(\omega-3k)^2/12\beta^2} + \dots, \quad (81)$$

where

$$S_1 = \frac{M}{\sqrt{2}\beta kd}, \quad S_3 = \frac{\sqrt{6}}{18} \frac{M^3}{\beta dr_0 M_\star^4}. \quad (82)$$

The first term in (81) represents the spectral line centered about $\omega = k$ that our detector would have seen without polymer quantization. The second term is the result of the leading order polymer correction: An additional spectral line centered about $\omega = 3k$. The relative amplitude of the two lines is

$$\frac{S_3}{S_1} \sim \frac{E_{\text{tot}} k \beta}{r_0 M_\star^4}, \quad (83)$$

where we have noted that the total energy in the pulse initially is

$$E_{\text{tot}} = \int \rho_{\text{KG}} d^3x \sim M^2/\beta, \quad (84)$$

assuming a long duration $k \gg \beta$ and a large initial radius $r_0\beta \gg 1$. Now, the initial pulse must have a size bigger than the Schwarzschild radius associated with E_{tot} ; i.e., $r_0 \gtrsim E_{\text{tot}}/M_{\text{pl}}^2$, so we obtain an upper bound on S_3/S_1 :

$$\frac{S_3}{S_1} \lesssim \left(\frac{E_{\text{KG}}}{E_{\text{pl}}} \right) \left(\frac{\Delta t}{t_{\text{pl}}} \right)^{-1} \left(\frac{M_\star}{M_{\text{pl}}} \right)^{-4} \sim 10^{-66} \left(\frac{E_{\text{KG}}}{10^5 \text{ eV}} \right) \left(\frac{\Delta t}{1 \text{ sec}} \right)^{-1} \left(\frac{M_\star}{M_{\text{pl}}} \right)^{-4}, \quad (85)$$

where Δt is the duration of the pulse observed on earth and E_{KG} is the energy per particle (photon) in the beam; for a GRB, one typically has $\Delta t \sim 1 \text{ sec}$ and $E_{\text{KG}} \sim 10^5 \text{ eV}$ [31]. As in the case for the LHC, we know of no evidence for exotic polymer effect in observed GRBs, which suggests that $S_3/S_1 \ll 1$. This yields another bound: $M_\star \gtrsim 1 \text{ TeV}$. This is virtually the same numeric result we obtained from the LHC.

V. DISCUSSION

We have developed an approach for studying polymer quantum field theory on a curved background spacetime. The quantization procedure is motivated by ideas used in loop quantum gravity. However, the representation used here is significantly different to the one used earlier [12]. The polymer quantization differs from the usual approach in two important aspects: the inner product is background independent, and there is a built-in mass (or length) scale M_\star due to the choice of basic observables. Using semi-classical methods we have derived an effective nonlinear wave equation

governing the field dynamics. The form of this semi-classical wave equation necessarily depends on the underlying quantum state, but we explicitly find that it reduces to the conventional Klein-Gordon equation at low energies.

In §III, we developed perturbative solutions to the nonlinear wave equation. The natural expansion parameter involves the amplitude of the lowest order contribution. We then specialized to waves with planar and spherical symmetry in §IIIB and §IIIC, respectively. For the plane-symmetric case, we found that the polymer corrections to the classical solution actually grow in time. This suggests that if one waits long enough, nonlinear effects will completely dominate the Klein-Gordon contribution to the wave.

The converse is true for spherical waves, which tend to decrease in amplitude as they travel outward from the center of symmetry. This implies that the polymer effects remain bounded as time progresses. This qualitative difference between planar and spherical wave highlights one of the most important features of the model: corrections to classical dynamics depend on more than just the frequency of the wave. Furthermore, if one begins with a spatially localized spherical wave-packet with characteristic wavenumber k , one obtains a whole series of wave-packets with wavenumber $k, 3k, 5k \dots$ at infinity.

In §IV, we derived some phenomenological implications of the effective wave equation. We found that the group velocity of planar wave-packets depends on their frequency, amplitude and shape. This is in sharp contrast to many models in the literature, which assume that the group velocity only receives frequency-dependent modifications. From these group velocities, we derive effective dispersion relations for the model. These bear some resemblance to the dispersion relations of weakly nonlinear water waves in that they depend explicitly on amplitude. Our method of extracting a dispersion relation appears to be new, and has potential applications to other nonlinear wave equations.

The fact that our perturbative solution breaks down after a finite amount of time has interesting practical implications, as discussed in §IVB. This implies that there is a “polymer coherence length” d_{poly} associated with plane-symmetric wave packets. After a wave packet has traveled a distance d_{poly} , polymer effects will completely dominate the classical behaviour. We express d_{poly} as a function of the frequency and luminosity of the beam. We calculate d_{poly} for the proton-antiproton beam in the Large Hadron Collider, and use the fact that no exotic behaviour has been reported in that device to deduce a lower limit of $M_\star \gtrsim 1 \text{ TeV}$. We also calculated d_{poly} for a highly collimated beam of cosmic rays, and found that it is many orders of magnitude larger than the size of the observable universe. Finally, we considered spherical gamma ray bursts in §IVC and demonstrated that if the characteristic frequency of the burst is $\omega = k$, an earth-based observer will see additional spectral lines centered at $\omega = (2n + 1)k$. From the amplitude ratio of the classical to polymer spectral features, we deduced that $M_\star \gtrsim 1 \text{ TeV}$ consistent with the LHC result mentioned above.

It is interesting to note that all of the modifications to wave propagation mentioned above are derived without including any quantum gravity effects, unlike other approaches such as doubly special relativity, loop quantum gravity, or non-commutative spacetime models. Another important feature worth emphasizing is that fundamental discreteness in this model leads to nonlinearities at short distances. This is contrary to the common assumption that the effects of fundamental discreteness manifest themselves via a linear wave equation with higher derivative terms.

There are a number of possible developments based on the ideas we have discussed, such as applications to Hawking radiation and to the spectrum of fluctuations in inflationary cosmology, which have as their basis a wave equation on a curved background. These are presently being studied. Another application is to homogeneous scalar field propagation on an FRW background. This work provides some novel results, including cosmological singularity avoidance and a mechanism for inflation [23]. The former result is a concrete realization of the intuition that in the Hamiltonian constraint, the compactification of the matter energy density that results from the semi-classical effective Hamiltonian leads to a bound on the gravitational terms in this constraint; the converse occurs in the LQC case, where it is the gravitational kinetic terms that are compactified by polymer geometry. We also show in [23] that the energy scale of polymer inflation is fixed by M_\star , which leads to an entirely different way of constraining the fundamental discreteness scale in our model.

Acknowledgments

This work was supported in part by the Natural Science and Engineering Research Council of Canada, and the Atlantic Association for Research in the Mathematical Sciences.

APPENDIX A: THE SEMI-CLASSICAL WAVE EQUATION ON A CURVED BACKGROUND

The method followed to derive an effective wave equation on Minkowski spacetime is applicable also to an arbitrary background. Here we restrict attention to metrics with lapse $N = 1$ and shift $N^a = 0$, and denote the spatial metric by q_{ab} and its determinant by q .

Let $\mathcal{H}(P_\phi(x), \phi(x))$ be the Hamiltonian density and define the effective Hamiltonian density \mathcal{H}^{eff} by the expectation value

$$\mathcal{H}^{\text{eff}}(P_0(x_k), \phi_0(x_k); \sigma, \lambda) \equiv \frac{1}{2} \langle P_0, \phi_0 | \frac{(\hat{P}_\phi^\lambda)^2}{\sqrt{q}} + \sqrt{q} \widehat{\nabla^2 \phi} | P_0, \phi_0 \rangle_{x_k}^{\sigma, \lambda} . \quad (\text{A1})$$

The semi-classical states given above have to be modified to take into account the density weights, and the fact that the peaking value for the scalar field is actually for ϕ_f rather than ϕ . With these changes, to leading order in the width σ , the kinetic part of the operator gives

$$\begin{aligned} \langle P_0, \phi_0 | (\hat{P}_\phi^\lambda)^2 | P_0, \phi_0 \rangle_{x_k}^{\sigma, \lambda} &= \frac{q}{\lambda^2} \langle P_0, \phi_0 | \left(2 - \hat{U}_{2\lambda}(P_\phi(x_k)) - \hat{U}_{2\lambda}^\dagger(P_\phi(x_k)) \right) | P_0, \phi_0 \rangle_{x_k}^{\sigma, \lambda} \\ &= \frac{q}{\lambda^2} \sin^2 \left(\frac{P_0 \lambda}{\sqrt{q}} \right) + \mathcal{O} \left(\frac{1}{\sigma} \right), \end{aligned} \quad (\text{A2})$$

where the last term represents the width corrections of the semi-classical states. The scalar gradient operator is diagonal in the basis, and as for the flat space case, its discrete aspect can be ignored for wavelengths large compared to the spatial lattice spacing ϵ . Thus we can write

$$\mathcal{H}^{\text{eff}}(x) = \frac{\sqrt{q}}{2\lambda^2} \sin^2 \left(\frac{P\lambda}{\sqrt{q}} \right) + \frac{\sqrt{q}}{2} \nabla^2 \phi. \quad (\text{A3})$$

This Hamiltonian density leads to the modified wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \left[1 - 4\lambda^2 \left(\frac{\partial \phi}{\partial t} \right)^2 \right]^{1/2} \left[\frac{1}{\sqrt{q}} \nabla^2 \phi - \frac{1}{4\lambda q} \left(\frac{\partial q}{\partial t} \right) \arcsin \left(2\lambda \frac{\partial \phi}{\partial t} \right) \right] = 0. \quad (\text{A4})$$

-
- [1] R. Gambini, J. Pullin, Phys. Rev. D59, 124021 (1999); J. Alfaro, Hugo A. Morales-Tecotl, Luis F. Urrutia, Phys. Rev. D65 (2002) 103509 (arXiv:hep-th/0108061); H. Sahlmann, T. Thiemann, Class. Quant. Grav. 23 (2006) 909-954 (arXiv:gr-qc/0207031); C. N. Kozameh, M.F. Parisi Class. Quant. Grav. 21 (2004) 2617-2621 (arXiv:gr-qc/0310014).
 - [2] Robert C. Myers, M. Pospelov, Phys. Rev. Lett. 90, 211601 (2003) (hep-ph/0301124).
 - [3] J. Collins, A. Perez, D. Sudarsky, L. Urrutia, H. Vucetich Phys. Rev. Lett. 93: 191301 (2004); e-Print: gr-qc/0403053.
 - [4] W. G. Unruh, Phys. Rev. D51, 2827-2838 (1995).
 - [5] S. Corley, T. Jacobson, Phys. Rev. D54, 1568 (1996).
 - [6] R. H. Brandenberger, J. Martin, Phys. Rev. D63, 123501 (2001).
 - [7] G. Amelino-Camelia, John Ellis, N.E. Mavromatos, D.V. Nanopoulos, Subir Sarkar, *Potential Sensitivity of Gamma-Ray Burst Observations to Wave Dispersion in Vacuo*, Nature 393, 763 (1998).
 - [8] Kenneth Greisen, Phys. Rev. Lett. 16, 748 (1966); G.T. Zatsepin and V.A. Kuzmin, JETP Lett. 41 (1966) 78.
 - [9] T. Jacobson, S. Liberati, D. Mattingly, *Quantum gravity phenomenology and Lorentz violation*, Springer Proc. Phys. 98, 83 (2005) (gr-qc/0404067); Annals Phys. 32, 150(2006) (astro-ph/0505267).
 - [10] T. Thiemann, "QSD V: Quantum gravity as the natural regulator of matter quantum field theories," Class. Quant. Grav. **15**, 1281 (1998) [arXiv:gr-qc/9705019].
 - [11] T. Thiemann, Class. Quant. Grav. 15, 839 (1998).
 - [12] A. Ashtekar, J. Lewandowski, H. Sahlmann, Class. Quant. Grav. 20 (2003) L11-1, (arXiv:gr-qc/0211012).
 - [13] V. Husain, O. Winkler, Class. Quant. Grav. 22, L 127 (2005); Class. Quant. Grav. 22, L135 (2005).
 - [14] T. Thiemann, "Introduction to modern canonical quantum general relativity," arXiv:gr-qc/0110034.
 - [15] A. Ashtekar, J. Lewandowski, "Background independent quantum gravity: A status report," Class. Quant. Grav. **21**, R53 (2004) [arXiv:gr-qc/0404018].
 - [16] L. Smolin, "An invitation to loop quantum gravity," arXiv:hep-th/0408048.
 - [17] C. Rovelli, "Loop quantum gravity," Living Rev. Rel. **11** (2008) 5.
 - [18] A. Ashtekar, S. Fairhurst and J. L. Willis, "Quantum gravity, shadow states, and quantum mechanics," Class. Quant. Grav. **20**, 1031 (2003) [arXiv:gr-qc/0207106].
 - [19] H. Halvorson, "Complementarity of representations in quantum mechanics", Stud. Hist. Phil. Mod. Phys., 35 (2004); arXiv: quant-ph/0110102.
 - [20] V. Husain, O. Winkler, Phys. Rev. D75, 024014 (2007). e-Print: gr-qc/0607097.
 - [21] A. Ashtekar, T. Pawłowski and P. Singh, "Quantum nature of the big bang: Improved dynamics," Phys. Rev. D **74**, 084003 (2006) [arXiv:gr-qc/0607039].
 - [22] M. Bojowald, "Loop quantum cosmology," Living Rev. Rel. **11**, 4 (2008).

- [23] G. M. Hossain, V. Husain and S. S. Seahra, arXiv:0906.2798 [astro-ph.CO].
- [24] M. Bojowald, “Absence of singularity in loop quantum cosmology,” Phys. Rev. Lett. **86**, 5227 (2001) [arXiv:gr-qc/0102069].
- [25] G. Date and G. M. Hossain, “Genericness of big bounce in isotropic loop quantum cosmology,” Phys. Rev. Lett. **94**, 011302 (2005) [arXiv:gr-qc/0407074].
- [26] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002) [arXiv:gr-qc/0012051].
- [27] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002) [arXiv:hep-th/0112090].
- [28] A. M. Kamchatnov, *Nonlinear periodic wave and their modulations: An introductory course* (World Scientific, 2000)
- [29] O. S. Bruning and P. Collier, Nature **448**, 285 (2007).
- [30] J. Abraham *et al.* [Pierre Auger Collaboration], Phys. Rev. Lett. **101**, 061101 (2008) [arXiv:0806.4302 [astro-ph]].
- [31] T. Piran, Rev. Mod. Phys. **76**, 1143 (2004) [arXiv:astro-ph/0405503].